Improved conformal mapping of the Borel plane

Ulrich D Jentschura and Gerhard Soff

Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany

E-mail: jentschura@physik.tu-dresden.de and soff@physik.tu-dresden.de

Abstract
The conformal mapping of the Borel plane can be utilized for the analytic continuation of the Borel transform to the entire positive real semi-axis and is thus helpful in the resummation of divergent perturbation series in quantum field theory. We observe that the convergence can be accelerated by the application of Padé approximants to the Borel transform expressed as a function of the conformal variable, i.e. by a combination of the analytic continuation via conformal mapping and a subsequent numerical approximation by rational approximants. The method is primarily useful in those cases where the leading (but not sub-leading) large-order asymptotics of the perturbative coefficients are known.

PACS numbers: 1115B, 1110J

The problem of the resummation of quantum field theoretic series is of obvious importance in view of the divergent, asymptotic character of the perturbative expansions [1–3]. The convergence can be accelerated when additional information is available about large-order asymptotics of the perturbative coefficients [4]. In the example cases discussed in [4], the location of several poles in the Borel plane, known from the leading and next-to-leading large-order asymptotics of the perturbative coefficients, is utilized in order to construct specialized resummation prescriptions. Here, we consider a particular perturbation series, investigated in [5], where only the *leading* large-order asymptotics of the perturbative coefficients are known to sufficient accuracy, and the subleading asymptotics have—not yet—been determined. Therefore, the location of only a single pole—the one closest to the origin—in the Borel plane is available. In this case, as discussed in [6,7], the (asymptotically optimal) conformal mapping of the Borel plane is an attractive method for the analytic continuation of the Borel transform beyond its circle of convergence and, to a certain extent, for accelerating the convergence of the Borel transforms. Here, we argue that the convergence of the transformation can be accelerated further when the Borel transforms, expressed as a function of the conformal variable which mediates the analytic continuation, are additionally convergence-accelerated by the application of Padé approximants.

First we discuss, in general terms, the construction of the improved conformal mapping of the Borel plane which is used for the resummation of the perturbation series defined in

equations (14) and (18) below. The method uses as input data the numerical values of a finite number of perturbative coefficients and the leading large-order asymptotics of the perturbative coefficients, which can, under appropriate circumstances, be derived from an empirical investigation of a finite number of coefficients, as has been done in [5]. We start from an asymptotic, divergent perturbative expansion of a physical observable $f(g)$ in powers of a coupling parameter g,

$$f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \tag{1}$$

and we consider the generalized Borel transform of the $(1, \lambda)$-type (see equation (4) in [4]),

$$f_{\mathrm{B}}^{(\lambda)}(u) \equiv f_{\mathrm{B}}^{(1,\lambda)}(u) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+\lambda)} \, u^n. \tag{2}$$

The full physical solution can be reconstructed from the divergent series (1) by evaluating the Laplace–Borel integral, which is defined as

$$f(g) = \frac{1}{g^{\lambda}} \int_0^{\infty} \mathrm{d}u \, u^{\lambda-1} \exp\left(-u/g\right) f_{\mathrm{B}}^{(\lambda)}(u). \tag{3}$$

The integration variable u is referred to as the Borel variable. The integration is carried out either along the real axis or infinitesimally above or below it (if Padé approximants are used for the analytic continuation, modified integration contours have been proposed [8]). The most prominent issue in the theory of the Borel resummation is the construction of an analytic continuation for the Borel transform (2) from a finite-order partial sum of the perturbation series (1), which we denote by

$$f_{\mathrm{B}}^{(\lambda),m}(u) = \sum_{n=0}^{m} \frac{c_n}{\Gamma(n+\lambda)} \, u^n. \tag{4}$$

The analytic continuation can be accomplished using the direct application of Padé approximants to the partial sums of the Borel transform $f_{\mathrm{B}}^{(\lambda),m}(u)$ [5, 8–10] or by a conformal mapping [6, 7, 11–13]. We now assume that the *leading* large-order asymptotics of the perturbative coefficients c_n defined in equation (1) is factorial, and that the coefficients display an alternating sign pattern. This indicates the existence of a singularity (branch point) along the negative real axis corresponding to the leading large-order growth of the perturbative coefficients, which we assume to be at $u = -1$. For Borel transforms which have only a single cut in the complex plane which extends from $u = -1$ to $u = -\infty$, the following conformal mapping has been recommended as optimal [6]:

$$z = z(u) = \frac{\sqrt{1+u} - 1}{\sqrt{1+u} + 1}. \tag{5}$$

Here, z is referred to as the conformal variable. The cut Borel plane is mapped onto the unit circle by the conformal mapping (5). We briefly mention that a large variety of similar conformal mappings have been discussed in the literature [14–19].

It is worth noting that conformal mappings which are adopted for doubly-cut Borel planes have been discussed in [6, 7]. We do not claim here that it would be impossible to construct conformal mappings which reflect the position of more than two renormalon poles or branch points in the complex plane. However, we stress that such a conformal mapping is likely to have a more complicated mathematical structure than, for example, the mapping defined in equation (27) in [6]. Using the alternative methods described in [4], poles (branch points) in the Borel plane corresponding to the subleading asymptotics can be incorporated easily provided their

position in the Borel plane is known. In a concrete example (see table 1 in [4]), 14 poles in the Borel plane have been fixed in the denominator of the Padé approximant constructed according to equations (53)–(55) in [4], and accelerated convergence of the transforms is observed. In contrast to the investigation [4], we assume here that only the *leading* large-order factorial asymptotics of the perturbative coefficients are known.

We continue with the discussion of the conformal mapping (5). It should be noted that for series whose leading singularity in the Borel plane is at $u = -u_0$ with $u_0 > 0$, an appropriate rescaling of the Borel variable $u \to |u_0| u$ is necessary on the right-hand side of equation (3). Then, $f_B^{(\lambda)}(|u_0| u)$ as a function of u has its leading singularity at $u = -1$ (see also equation (41.57) in [2]). The Borel integration variable u can be expressed as a function of z as follows:

$$u(z) = \frac{4z}{(z-1)^2}. \tag{6}$$

The mth partial sum of the Borel transform (4) can be rewritten, upon expansion of the u in powers of z, as

$$f_B^{(\lambda),m}(u) = f_B^{(\lambda),m}\big(u(z)\big) = \sum_{n=0}^{m} C_n z^n + \mathcal{O}(z^{m+1}) \tag{7}$$

where the coefficients C_n as a function of the c_n are uniquely determined (see, e.g., equations (36) and (37) of [6]). We define the partial sum of the Borel transform, expressed as a function of the conformal variable z, as

$$f_B^{\prime(\lambda),m}(z) = \sum_{n=0}^{m} C_n z^n. \tag{8}$$

In a previous investigation [6], Caprini and Fischer evaluate the following transforms:

$$\mathcal{T}_m' f(g) = \frac{1}{g^\lambda} \int_0^\infty \mathrm{d}u\, u^{\lambda-1} \exp(-u/g)\, f_B^{\prime(\lambda),m}(z(u)). \tag{9}$$

Caprini and Fischer [6] observe the apparent numerical convergence with increasing m. The limit as $m \to \infty$, provided it exists, is then assumed to represent the complete, physically relevant solution,

$$f(g) = \lim_{m\to\infty} \mathcal{T}_m' f(g). \tag{10}$$

We do not consider the question of the existence of this limit here (for an outline of questions related to these issues we refer to [7]).

In the absence of further information on the analyticity domain of the Borel transform (2), we cannot necessarily conclude that $f_B^{(\lambda)}(u(z))$ as a function of z is analytic inside the unit circle of the complex z-plane, or that, for example, the conditions of Theorem 5.2.1 of [20] are fulfilled. Therefore, we propose a modification of the transforms (9). In particular, we advocate the evaluation of (lower-diagonal) Padé approximants [20, 21] to the function $f_B^{\prime(\lambda),m}(z)$, expressed as a function of z,

$$f_B^{\prime\prime(\lambda),m}(z) = \left[\frac{[\![m/2]\!]}{[\![(m+1)/2]\!]} \right]_{f_B^{\prime(\lambda),m}}(z). \tag{11}$$

We define the transforms,

$$\mathcal{T}_m'' f(g) = \frac{1}{g^\lambda} \int_{C_j} \mathrm{d}u\, u^{\lambda-1} \exp(-u/g)\, f_B^{\prime\prime(\lambda),m}\big(z(u)\big) \tag{12}$$

where the integration contours C_j ($j = -1, 0, 1$) have been defined in [8]. These integration contours have been shown to provide the physically correct analytic continuation of resummed

Table 1. Resummation of the perturbation series (14) for the anomalous dimension γ function of the six-dimensional ϕ^3 theory by the method defined in equations (1)–(12). The transforms $\mathcal{T}_m'' \gamma_{\text{hopf}}(g)$ are shown in the transformation order $m = 28,\ 29,\ 30$. The coupling g assumes the values $g = 5.0,\ 5.5,\ 6.0$ and $g = 10.0$.

m	$\mathcal{T}_m'' \gamma_{\text{hopf}}(5.0)$	$\mathcal{T}_m'' \gamma_{\text{hopf}}(5.5)$	$\mathcal{T}_m'' \gamma_{\text{hopf}}(6.0)$	$\mathcal{T}_m'' \gamma_{\text{hopf}}(10.0)$
28	−0.501 565 232	−0.538 352 234	−0.573 969 740	−0.827 506 173
29	−0.501 565 232	−0.538 352 233	−0.573 969 738	−0.827 506 143
30	−0.501 565 231	−0.538 352 233	−0.573 969 738	−0.827 506 136

Table 2. Resummation of the perturbation series (18) for the anomalous dimension of the Yukawa coupling via the method defined in equations (1)–(12). The transforms $\mathcal{T}_m'' \tilde{\gamma}_{\text{hopf}}(g)$ are shown in the order of transformation $m = 28, 29, 30$. For the coupling g, we consider the values $g = 5.0, 5.5, 6.0$ and $g = 30^2/(4\pi)^2 = 5.69932\ldots$.

m	$\mathcal{T}_m'' \tilde{\gamma}_{\text{hopf}}(5.0)$	$\mathcal{T}_m'' \tilde{\gamma}_{\text{hopf}}(5.5)$	$\mathcal{T}_m'' \tilde{\gamma}_{\text{hopf}}(6.0)$	$\mathcal{T}_m'' \tilde{\gamma}_{\text{hopf}}(30^2/(4\pi)^2)$
28	−1.669 071 213	−1.800 550 588	−1.928 740 624	−1.852 027 809
29	−1.669 071 214	−1.800 550 589	−1.928 740 626	−1.852 027 810
30	−1.669 071 214	−1.800 550 589	−1.928 740 625	−1.852 027 810

perturbation series for those cases where the evaluation of the standard Laplace–Borel integral (3) is impossible due to an insufficient analyticity domain of the integrand (possibly due to multiple branch cuts) or due to spurious singularities in view of the finite order of the Padé approximations defined in (11). We should mention potential complications due to multi-instanton contributions, as discussed for example in ch 43 of [2] (these are not encountered in the current investigation). In this paper, we use exclusively the contour C_0 which is defined as the half sum of the contours C_{-1} and C_{+1} displayed in figure 1 in [8]. At increasing m, the limit as $m \to \infty$, provided it exists, is then again assumed to represent the complete, physically relevant solution,

$$f(g) = \lim_{m \to \infty} \mathcal{T}_m'' f(g). \tag{13}$$

Because we take advantage of the special integration contours C_j, analyticity of the Borel transform $f_B^{(\lambda)}(u(z))$ inside the unit circle of the complex z-plane is not required, and additional acceleration of the convergence is mediated by employing Padé approximants in the conformal variable z.

We consider the resummation of two particular perturbation series discussed in [5] for the anomalous dimension γ function of the ϕ^3 theory in six dimensions and the Yukawa coupling in four dimensions. The perturbation series for the ϕ^3 theory is given in equation (16) in [5],

$$\gamma_{\text{hopf}}(g) \sim \sum_{n=1}^{\infty} (-1)^n \frac{G_n}{6^{2n-1}} g^n \tag{14}$$

where the coefficients G_n are given in table 1 in [5] for $n = 1, \ldots, 30$ (the G_n are real and positive). We denote the coupling parameter a used in [5] as g; this is done in order to ensure compatibility with the general power series given in equation (1). Empirically, Broadhurst and Kreimer [5] derive the large-order asymptotics

$$G_n \sim \text{constant} \times 12^{n-1}\, \Gamma(n+2) \qquad n \to \infty \tag{15}$$

by investigating the explicit numerical values of the coefficients $G_1, \ldots, G_{30}$. The leading asymptotics of the perturbative coefficients c_n are therefore (up to a constant prefactor)

$$c_n \sim (-1)^n \frac{\Gamma(n+2)}{3^n} \qquad n \to \infty. \tag{16}$$

This implies that the λ-parameter in the Borel transform (2) should be set to $\lambda = 2$ (see also the notion of an asymptotically optimized Borel transform discussed in [4]). In view of equation (16), the pole closest to the origin of the Borel transform (2) is expected at

$$u = u_0^{\text{hopf}} = -3 \tag{17}$$

and a rescaling of the Borel variable $u \to 3u$ in equation (3) then leads to an expression to which the method defined in equations (1)–(12) can be applied directly. For the Yukawa coupling, the γ-function reads

$$\tilde{\gamma}_{\text{hopf}}(g) \sim \sum_{n=1}^{\infty} (-1)^n \frac{\tilde{G}_n}{2^{2n-1}} g^n \tag{18}$$

where the $\tilde{G}_n$ are given in table 2 in [5] for $n = 1, \ldots, 30$. Empirically, i.e. from an investigation of the numerical values of $\tilde{G}_1, \ldots, \tilde{G}_{30}$, the following factorial growth in large order is derived [5]:

$$\tilde{G}_n \sim \text{constant}' \times 2^{n-1} \Gamma(n + 1/2) \qquad n \to \infty. \tag{19}$$

This leads to the following asymptotics for the perturbative coefficients (up to a constant prefactor),

$$c_n \sim (-1)^n \frac{\Gamma(n + 1/2)}{2^n} \qquad n \to \infty. \tag{20}$$

This implies that an asymptotically optimal choice [4] for the λ-parameter in (2) is $\lambda = 1/2$. The first pole of the Borel transform (2) is therefore expected at

$$u = \tilde{u}_0^{\text{hopf}} = -2. \tag{21}$$

A rescaling of the Borel variable according to $u \to 2u$ in (3) enables the application of the resummation method defined in equations (1)–(12).

In table 1, numerical values for the transforms $\mathcal{T}_m'' \gamma_{\text{hopf}}(g)$ are given, which have been evaluated according to equation (12). The transformation order is in the range $m = 28, 29, 30$, and we consider coupling parameters $g = 5.0, 5.5, 6.0$ and $g = 10.0$. The numerical values of the transforms display apparent convergence to about nine significant figures for $g \leqslant 6.0$ and to about seven figures for $g = 10.0$. In table 2, numerical values for the transforms $\mathcal{T}_m'' \tilde{\gamma}_{\text{hopf}}(g)$ calculated according to equation (12) are shown in the range $m = 28, 29, 30$ for (large) coupling strengths $g = 5.0, 5.5, 6.0$. Additionally, the value $g = 30^2/(4\pi)^2 = 5.69932\ldots$ is considered as a special case (as was done in [5]). Again, the numerical values of the transforms display apparent convergence to about nine significant figures. At large coupling $g = 12.0$, the apparent convergence of the transforms suggests the following values: $\gamma_{\text{hopf}}(12.0) = -0.939\,114\,3(2)$ and $\tilde{\gamma}_{\text{hopf}}(12.0) = -3.287\,176\,9(2)$. The numerical results for the Yukawa case, i.e. for the function $\tilde{\gamma}_{\text{hopf}}$, have recently been confirmed by an improved analytic, nonperturbative investigation [22] which extends the perturbative calculation [5].

We note that the transforms $\mathcal{T}_m' \gamma_{\text{hopf}}(g)$ and $\mathcal{T}_m' \tilde{\gamma}_{\text{hopf}}(g)$ calculated according to equation (9), i.e. by the unmodified conformal mapping, typically exhibit apparent convergence to five or six significant figures in the transformation order $m = 28, 29, 30$ and at large coupling $g \geqslant 5$. Specifically, the numerical values for $g = 5.0$ are:

$$\mathcal{T}_{28}' \gamma_{\text{hopf}}(g = 5.0) = -0.501\,567\,294$$

$$\mathcal{T}_{29}' \gamma_{\text{hopf}}(g = 5.0) = -0.501\,564\,509$$

$$\mathcal{T}_{30}' \gamma_{\text{hopf}}(g = 5.0) = -0.501\,563\,626.$$

These results, when compared to the data in table 1, exemplify the acceleration of the convergence by the additional Padé approximation of the Borel transform *expressed as a function of the conformal variable* (see equation (11)).

It is not claimed here that the resummation method defined in equations (1)–(12) necessarily provides the fastest possible rate of convergence for the perturbation series defined in equations (14) and (18). Further improvements should be feasible, especially if particular properties of the input series are known and exploited (see in part the methods described in [4]). We also note possible improvements based on a large-coupling expansion [23], in particular for excessively large values of the coupling parameter g, or methods based on order-dependent mappings (see [11, 12] or the discussion following equation (41.67) in [2]).

The conformal mapping [6, 7] is capable of accomplishing the analytic continuation of the Borel transform (2) beyond the circle of convergence. Padé approximants, applied directly to the partial sums of the Borel transform (4), provide an alternative to this method [4, 5, 8–10]. Improved rates of convergence can be achieved when the convergence of the transforms obtained by conformal mapping in equation (7) is accelerated by evaluating Padé approximants as in equation (11), and conditions on analyticity domains can be relaxed in a favourable way when these methods are combined with the integration contours from [8]. Numerical results for the resummed values of the perturbation series (14) and (18) are provided in tables 1 and 2. By the improved conformal mapping and other optimized resummation techniques (see, e.g., the methods introduced in [4]) the applicability of perturbative (small-coupling) expansions can be generalized to the regime of large coupling and still lead to results of relatively high accuracy.

Acknowledgments

References

[1]   LeGuillou J C and Zinn-Justin J 1990 *Large-Order Behaviour of Perturbation Theory* (Amsterdam: North-Holland)
[2]   Zinn-Justin J 1996 *Quantum Field Theory and Critical Phenomena* 3rd edn (Oxford: Clarendon)
[3]   J Fischer 1997 *Int. J. Mod. Phys.* A **12** 3625
[4]   Jentschura U D, Weniger E and Soff G 2000 *J. Phys. G: Nucl. Phys.* **26** 1545
       (Jentschura U D, Weniger E and Soff G 2000 *Preprint* Los Alamos hep-ph/0005198)
[5]   Broadhurst D and Kreimer D 2000 *Phys. Lett.* B **475** 63
[6]   Caprini I and Fischer J 1999 *Phys. Rev.* D **60** 054014
[7]   Caprini I and Fischer J 2000 *Phys. Rev.* D **62** 054007
       (Caprini I and Fischer J 2000 *Preprint* Los Alamos hep-ph/0002016)
[8]   Jentschura U D 2000 *Phys. Rev.* D **62** 076001
       (Jentschura U D 2000 *Preprint* Los Alamos hep-ph/0001135)
[9]   Rączka P A 1991 *Phys. Rev.* D **43** R9
[10] Pindor M 1999 Padé approximants and Borel summation for QCD perturbation series *Preprint* Los Alamos hep-th/9903151
[11] Seznec R and Zinn-Justin J 1979 *J. Math. Phys.* **20** 1398
[12] LeGuillou J C and Zinn-Justin J 1983 *Ann. Phys., NY* **147** 57
[13] Guida R, Konishi K and Suzuki H 1995 *Ann. Phys., NY* **241** 152
[14] Broadhurst D J, Baikov P A, Ilyin V A, Fleischer J, Tarasov O V and Smirnov V A 1994 *Phys. Lett.* B **329** 103
[15] Altarelli G, Nason P and Ridolfi G 1995 *Z. Phys.* C **68** 257
[16] Soper D E and Surguladze L R 1996 *Phys. Rev.* D **54** 4566
[17] Chetyrkin K G, Kühn J H and Steinhauser M 1996 *Phys. Lett.* B **371** 93

[18] Chetyrkin K G, Kühn J H and Steinhauser M 1996 *Nucl. Phys.* B **482** 213
[19] Chetyrkin K G, Harlander R and Steinhauser M 1998 *Phys. Rev.* D **58** 014012
[20] Baker G A and Graves-Morris P 1996 *Padé Approximants* 2nd edn (Cambridge: Cambridge University Press)
[21] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill)
[22] Broadhurst D and Kreimer D 2000 Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality *Preprint* Los Alamos hep-th/0012146
[23] Weniger E J 1996 *Phys. Rev. Lett.* **77** 2859